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On Simple Components of Cocommutative Hopf Algebras

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Let k be a field. Let K be a finite cyclic extension of k with Galois group $\pi = \langle\sigma\rangle$ and let $n = [K : k]$. Given $\alpha \in k - \{0\}$ we define

$$\alpha_{\sigma^i,\sigma^j} := \begin{cases} 1 & \text{when } i + j < n \\ \alpha & \text{when } i + j \geqslant n. \end{cases}$$

The crossed product of K and π by the factor set $\{\alpha_{\sigma^i,\sigma^j}\}_{0 \leqslant i,j < n}$ is called a cyclic algebra over k and is denoted by (K, α).

The purpose of this paper is to prove

PROPOSITION 2.1. *Let k be a field of characteristic 0. Then any cyclic k-algebra is realizable as a simple component of a finite-dimensional cocommutative k-Hopf algebra.*

In particular, if k is an algebraic number field, then, by the famous Noether–Brauer–Hasse theorem (e.g., [1]), any finite-dimensional central simple k-algebra is a cyclic k-algebra. Hence from Proposition 2.1 we get the following remarkable

THEOREM. *Let k be an algebraic number field. Then any finite-dimensional central simple k-algebra is realizable as a simple component of a finite-dimensional cocommutative k-Hopf algebra.*

However, the following problem is kept open.

Problem. Let k be an arbitrary field of characteristic 0. Is a finite-dimensional central simple k-algebra realizable as a simple component of a finite-dimensional cocommutative k-Hopf algebra?

Let k be a field of characteristic 0. Let H be a finite-dimensional cocommutative k-Hopf algebra. It follows immediately from Kostant's theorem (e.g., [2]) that there exist a finite Galois extension K of k and a finite group G such that $H \otimes_k K \cong KG$ as K-Hopf algebras. (More generally, in [4] Larson showed that any finite-dimensional involutory k-Hopf algebra is semisimple. This

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means that H and H^* are semisimple.) Therefore a finite-dimensional cocommutative k-Hopf algebra can be obtained from a group algebra by Galois descent. It should be noted (e.g., [3]) that there exists a finite-dimensional central simple algebra over an algebraic number field k which is not realizable as a simple component of a group algebra over k.

Now let π, G be finite groups and let $\Psi'(\pi, G) = \{\psi \mid \psi$ is a group homomorphism of π to Aut $G\}$. For $\psi_1, \psi_2 \in \Psi'(\pi, G)$ we define $\psi_1 \sim \psi_2$ if there exists an element τ of Aut G such that $\tau^{-1}\psi_1(\sigma)\tau = \psi_2(\sigma)$ for any $\sigma \in \pi$. Then this is obviously an equivalence relation in $\Psi'(\pi, G)$. We denote by $\Psi(\pi, G)$ the set of all equivalence classes in $\Psi'(\pi, G)$.

Further let k be a field and let K be a finite Galois extension of k with Galois group $\cong \pi$. Let $H'(K/k, G) = \{H \mid H$ is a k-Hopf algebra such that $H \otimes_k K \cong KG$ as K-Hopf algebras$\}$ and denote by $H(K/k, G)$ the set of all isomorphism types as k-Hopf algebras in $H'(K/k, G)$. For $\psi \in \Psi'(\pi, G)$, $\sigma \in \pi$, we can regard $\psi(\sigma)$ as a k-automorphism of KG by

$$\psi(\sigma)\Big(\sum_{g\in G} a_g g\Big) = \sum_{g\in G} \sigma(a_g)\, \psi(\sigma)g \qquad \text{for} \quad a_g \in K.$$

We define the map ϕ: $\Psi'(\pi, G) \to H'(K/k, G)$ by $\phi(\psi) = KG^{\psi(\pi)}$.

Another main result in this paper is

THEOREM 1.4. *The map* ϕ: $\Psi'(\pi, G) \to H'(K/k, G)$ *induces the bijection* $\bar{\phi}$: $\Psi(\pi, G) \to H(K/k, G)$.

In this paper we use the same notation and terminology as in [2].

1. Galois Descent

We begin with

PROPOSITION 1.1 (Kostant). *Let k be a field of characteristic* 0. *Let H be a finite-dimensional cocommutative k-Hopf algebra. Then there exist a finite Galois extension K of k and a finite group G such that $H \otimes_k K \cong KG$ as K-Hopf algebras.*

Proof. Let $\bar{k}$ be an algebraic closure of k. Let $G = \{g \in H \otimes_k \bar{k} \mid (\Delta \otimes 1)(g) = g \otimes g\}$ and $L = \{l \in H \otimes_k \bar{k} \mid (\Delta \otimes 1)(l) = l \otimes 1 + 1 \otimes l\}$. Then G is a group and the elements of G are linearly independent over $\bar{k}$ [2, (3.2.1)]. L can be regarded as a Lie algebra under $[l_1, l_2] = l_1 l_2 - l_2 l_1$. We denote by $U(L)$ the universal enveloping algebra of L. By Kostant's theorem [2, (8.1.5) and (13.0.1)] we have $H \otimes_k \bar{k} \cong U(L) \otimes_{\bar{k}} \bar{k}G$ as $\bar{k}$-vector spaces. If $L \neq 0$, then $\dim_{\bar{k}} U(L) = \infty$, which contradicts the assumption that $\dim_k H < \infty$. Hence $H \otimes_k \bar{k} \cong \bar{k}G$ as $\bar{k}$-vector spaces. Since $\bar{k}G$ is a sub-Hopf algebra of $H \otimes_k \bar{k}$, we have $H \otimes_k \bar{k} \cong \bar{k}G$ as $\bar{k}$-Hopf algebras. It is seen easily that $(\bar{k}G)^* \cong \bar{k} \oplus \cdots \oplus \bar{k}$ as $\bar{k}$-algebras. Therefore H^* is a commutative semisimple algebra. Hence we have

$H^* \simeq K_1 \oplus \cdots \oplus K_t$ for some finite extensions K_i of k. Let K be a finite Galois extension of k containing all K_i, $i = 1,\ldots, t$. Then $(H \otimes_k K)^* \simeq H^* \otimes_k K \simeq K \oplus \cdots \oplus K$ as K-algebra. Let $e_1, \ldots, e_n$ be the orthogonal idempotents of $H^* \otimes_k K$. And let $\{g_i\}$ be the dual basis of $\{e_i\}$. Then we easily can check that $\Delta(g_i) = g_i \otimes g_i$. Hence we have $\{g_i\} = G$ and $H \otimes_k K \simeq KG$ as K-Hopf algebras.

Remark. By using Larson's result [4, Theorem 4.3] we can prove this proposition more easily.

LEMMA 1.2. *Let k be a field, let K be a finite Galois extension of k, let $\pi = \mathrm{Gal}(K/k)$ be the Galois group of K over k, and let π_0 be a subgroup of π. Let $\{\tau_0\,(=1), \tau_1, \ldots, \tau_l\}$ be a set of (left) representatives of π/π_0 in π and let $\{t_0, t_1, \ldots, t_l\}$ be a k-basis of K^{π_0}. Then there exist $(l+1)^3$ elements α_{ij}^m, $0 \leqslant i, j, m \leqslant l$, of k such that*

$$\sum_{0 \leqslant i,j \leqslant l} \alpha_{ij}^m \tau_s(t_i)\, \tau_t(t_j) = \tau_s(t_m)\delta_{st}, \qquad \textit{for all } 0 \leqslant m, s, t \leqslant l.$$

Proof. Let $A = (\tau_i(t_j))_{i,j}$. Let A_{ij} be the $l \times l$ matrix obtained from A by deleting the ith row and jth column and $\Delta_{ij} = (-1)^{i+j} \det A_{ij}$. Since $\tau(\Delta_{0i}/\det A) = \Delta_{0i}/\det A$ for all $\tau \in \pi_0$, we have $\Delta_{0i}/\det A \in K^{\pi_0}$. Then there exist $(l+1)^3$ elements α_{ij}^m in k satisfying $(\Delta_{0i}/\det A)t_m = \sum_{j=0}^{l} \alpha_{ij}^m t_j$. We also have $\tau_t(\Delta_{0i}/\det A) = \Delta_{ti}/\det A$ for all $0 \leqslant t \leqslant l$, and so $\sum_{j=0}^{l} \alpha_{ij}^m \tau_t(t_j) = (\Delta_{ti}/\det A)\,\tau_t(t_m)$ for all $0 \leqslant t \leqslant l$. Therefore

$$\begin{aligned}
&\left(\sum_{j=0}^{l} \alpha_{ij}^m \tau_t(t_j)\right)_{i,t} \\
&\quad = (1/\det A)(\Delta_{ti}\tau_t(t_m))_{i,t} \\
&\quad = (1/\det A)\left(\sum_{s=0}^{l} \Delta_{si}\tau_s(t_m)\delta_{st}\right)_{i,t} \\
&\quad = (1/\det A)^t(\Delta_{ij})_{i,j}(\tau_s(t_m)\delta_{st})_{s,t}.
\end{aligned}$$

Thus we have $(\tau_s(t_i))_{s,i}(\sum_{j=0}^{l} \alpha_{ij}^m \tau_t(t_j))_{i,t} = (\tau_s(t_m)\,\delta_{st})_{s,t}$. This means that $\sum_{0 \leqslant i,j \leqslant l} \alpha_{ij}^m \tau_s(t_i)\,\tau_t(t_j) = \tau_s(t_m)\,\delta_{st}$.

PROPOSITION 1.3. *Let k be a field, let K be a finite Galois extension of k, and let G be a finite group. Let $\pi = \mathrm{Gal}(K/k)$. Then*

(1) *$KG^{\psi(\pi)}$ is a k-Hopf algebra, and $KG^{\psi(\pi)} \otimes_k K \simeq KG$ as K-Hopf algebras.*

(2) *$KG^{\psi_1(\pi)} \simeq KG^{\psi_2(\pi)}$ as k-Hopf algebras if and only if $\psi_1 \sim \psi_2$.*

Proof. Let $\psi \in \Psi(\pi, G)$. Let $G = D_1 \cup \cdots \cup D_m$ be the decomposition of G into $\psi(\pi)$-orbits D_i, $1 \leqslant i \leqslant m$. Given $g_i \in D_i$ we put $\pi_i = \{\sigma \in \pi \mid \psi(\sigma)(g_i) = g_i\}$.

Let $\{\tau_0^{(i)}\ (=1), \tau_1^{(i)}, \ldots, \tau_{l_i}^{(i)}\}$ be a set of representatives of π/π_i in π. Then we have $D_i = \{\psi(\tau)(g_i) \mid \tau \in \pi\} = \{\psi(\tau_0^{(i)})(g_i)\ (= g_i), \psi(\tau_1^{(i)})(g_i), \ldots, \psi(\tau_{l_i}^{(i)})(g_i)\}$. Since D_i is a $\psi(\pi)$-orbit, for $\sum_{g\in G} a_g g \in KG^{\psi(\pi)}$ we have

$$\sum_{g\in G} a_g g = \sum_{i=1}^{m} \sum_{j=0}^{l_i} \tau_j^{(i)}(a_{g_i})\, \psi(\tau_j^{(i)})(g_i),$$

and therefore $KG^{\psi(\pi)} = KD_1^{\psi(\pi)} \oplus \cdots \oplus KD_m^{\psi(\pi)}$ as k-vector spaces and $KD_i^{\psi(\pi)} = \{\sum_{j=0}^{l_i} \tau_j^{(i)}(a)\, \psi(\tau_j^{(i)})(g_i) \mid a \in K^{\pi_i}\}$, because $\tau(a_{g_i})\, \psi(\tau)(g_i) = \tau(a_{g_i}) g_i$ for all $\tau \in \pi_i$.

Let S be the antipode of KG. For $\sum_{g\in G} a_g g \in KG^{\psi(\pi)}$

$$\begin{aligned}
\sum_{g\in G} a_g g^{-1} &= S\Big(\sum_{g\in G} a_g g\Big) \\
&= S\Big(\sum_{g\in G} \tau(a_g)\, \psi(\tau)(g)\Big) \\
&= \sum_{g\in G} \tau(a_g)(\psi(\tau)(g))^{-1} \\
&= \sum_{g\in G} \tau(a_g)\, \psi(\tau)(g^{-1}),
\end{aligned}$$

and so $S(KG^{\psi(\pi)}) = KG^{\psi(\pi)}$.

Next we will prove that $\Delta(KD_i^{\psi(\pi)}) \subseteq KD_i^{\psi(\pi)} \otimes_k KD_i^{\psi(\pi)}$, where Δ is the comultiplication of KG. Let $\{t_0, \ldots, t_{l_i}\}$ be a k-basis of K^{π_i}. If we put

$$a_k = \sum_{j=0}^{l_i} \tau_j^{(i)}(t_k)\, \psi(\tau_j^{(i)})(g_i),$$

$k = 0, \ldots, l_i$, then $\{a_0, \ldots, a_{l_i}\}$ is a k-basis of $KD_i^{\psi(\pi)}$, and by (1.2) we can find $(l_i + 1)^3$ elements α_{ij}^m in k such that $\sum_{0\leqslant u,v\leqslant l_i} \alpha_{uv}^m \tau_s^{(i)}(t_u)\, \tau_t^{(i)}(t_v) = \tau_s^{(i)}(t_m)\, \delta_{st}$. Then we have

$$\begin{aligned}
\sum_{0\leqslant u,v\leqslant l_i} \alpha_{uv}^m a_u \otimes a_v
&= \sum_{u,v} \alpha_{uv}^m \sum_{s,t} (\tau_s^{(i)}(t_u)\, \psi(\tau_s^{(i)})(g_i) \otimes \tau_t^{(i)}(t_v)\, \psi(\tau_t^{(i)})(g_i)) \\
&= \sum_{s,t} \sum_{u,v} \alpha_{uv}^m \tau_s^{(i)}(t_u)\, \tau_t^{(i)}(t_v)(\psi(\tau_s^{(i)})(g_i) \otimes \psi(\tau_t^{(i)})(g_i)) \\
&= \sum_{s,t} \tau_s^{(i)}(t_m)\, \delta_{st}(\psi(\tau_s^{(i)})(g_i) \otimes \psi(\tau_t^{(i)})(g_i))
\end{aligned}$$

$$= \sum_s \tau_s^{(i)}(t_m)(\psi(\tau_s^{(i)})(g_i) \otimes \psi(\tau_s^{(i)})(g_i))$$

$$= \Delta \left(\sum_s \tau_s^{(i)}(t_m)\, \psi(\tau_s^{(i)})(g_i)\right)$$

$$= \Delta(a_m).$$

Therefore $\Delta(KD_i^{\psi(\pi)}) \subseteq KD_i^{\psi(\pi)} \otimes_k KD_i^{\psi(\pi)}$. Thus $KG^{\psi(\pi)}$ is a k-Hopf algebra satisfying $KG^{\psi(\pi)} \otimes_k K \simeq KG$ as a K-Hopf algebra.

We assume that $\psi_1 \sim \psi_2$. Let τ be an element of Aut G such that $\tau^{-1}\psi_1(\sigma)\tau = \psi_2(\sigma)$ for all $\sigma \in \pi$. We regard τ as a K-Hopf algebra automorphism of KG by $\tau(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \tau(g)$ for $a_g \in K$. Let $\sum_{g\in G} a_g g \in KG^{\psi_2(\pi)}$. Then for all $\sigma \in \pi$ we have

$$\sum_g \sigma(a_g)\, \psi_1(\sigma)\, \tau(g)$$

$$= \sum_g \sigma(a_g)\, \tau(\tau^{-1}\psi_1(\sigma)\, \tau(g))$$

$$= \sum_g \sigma(a_g)\, \tau(\psi_2(\sigma)(g))$$

$$= \tau \left(\sum_g \sigma(a_g)\, \psi_2(\sigma)(g)\right)$$

$$= \tau \left(\sum_g a_g g\right)$$

$$= \sum_g a_g \tau(g).$$

Therefore τ is a map of $KG^{\psi_2(\pi)}$ into $KG^{\psi_1(\pi)}$. Since τ is a bijection, we have $KG^{\psi_1(\pi)} \simeq KG^{\psi_2(\pi)}$ as k-Hopf algebras.

Conversely suppose that $KG^{\psi_1(\pi)} \simeq KG^{\psi_2(\pi)}$ as k-Hopf algebras and denote this isomorphism by ϕ. Let α_i; $KG^{\psi_i(\pi)} \otimes_k K \xrightarrow{\sim} KG$, $i = 1, 2$, be K-Hopf algebra isomorphisms which can be found by (1), and put $\eta = \alpha_2(\phi \otimes 1)\alpha_1^{-1}$. Since $\alpha_1, \alpha_2, \phi \otimes 1$ are K-Hopf algebra maps, η is a K-Hopf algebra map. Therefore $\Delta(\eta(g)) = (\eta \otimes \eta)(\Delta(g)) = \eta(g) \otimes \eta(g)$ for all $g \in G$, which implies that η is an automorphism of G. Since α_2 is an isomorphism, for any $g \in G$ we can find elements $h_i \in KG^{\psi_2(\pi)}$, $l_i \in K$ such that $\sum_i \alpha_2(h_i \otimes l_i) = g$. From this we have

$$\eta\psi_1(\sigma)\, \eta^{-1}(g)$$

$$= \sum_i \eta\psi_1(\sigma)\, \alpha_1(\phi^{-1} \otimes 1)(h_i \otimes l_i)$$

$$= \sum_i \eta\psi_1(\sigma)\, \alpha_1(\phi^{-1}(h_i) \otimes l_i)$$

$$= \sum_i \eta\psi_1(\sigma)(\phi^{-1}(h_i)l_i) = \sum_i \eta\phi^{-1}(h_i)\,\sigma(l_i)$$

$$= \sum_i \eta\alpha_1(\phi^{-1}(h_i) \otimes \sigma(l_i)) = \sum_i \alpha_2(h_i \otimes \sigma(l_i))$$

$$= \psi_2(\sigma)\left(\sum_i \alpha_2(h_i \otimes l_i)\right) = \psi_2(\sigma)(g).$$

This completes the proof of (2).

We now have

THEOREM 1.4. *Let k be a field, let K be a finite Galois extension of k, and let G be a finite group. Let $\pi = \mathrm{Gal}(K/k)$. Then the map ϕ; $\Psi'(\pi, G) \to H'(K/k, G)$ defined by $\phi(\psi) = KG^{\psi(\pi)}$ induces the bijection $\bar{\phi}$; $\Psi(\pi, G) \to H(K/k, G)$.*

Proof. By (1.3) $\bar{\phi}$ is well defined and injective. Hence it suffices to show that $\bar{\phi}$ is surjective. Let $H \in H'(K/k, G)$. Let α; $H \otimes_k K \simeq KG$ as K-Hopf algebras and $\psi(\sigma) = \alpha(1 \otimes \sigma)\alpha^{-1}$ for $\sigma \in \pi = \mathrm{Gal}(K/k)$. Then we see easily that $\psi(\sigma_1\sigma_2) = \psi(\sigma_1)\,\psi(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \pi$ and that $\alpha(H) = KG^{\psi(\pi)}$. We denote by Δ_H the comultiplication of H. Then $\Delta\alpha = (\alpha \otimes \alpha)(\Delta_H \otimes 1)$, because α is a K-Hopf algebra map. By using this we have, for all $g \in G$,

$$\begin{aligned}
\Delta(\psi(\sigma)(g)) &= \Delta(\alpha(1 \otimes \sigma)\,\alpha^{-1}(g)) \\
&= (\alpha \otimes \alpha)(\Delta_H \otimes 1)(1 \otimes \sigma)\,\alpha^{-1}(g) \\
&= (\alpha \otimes \alpha)((1 \otimes \sigma) \otimes (1 \otimes \sigma))(\Delta_H \otimes 1)\,\alpha^{-1}(g) \\
&= (\alpha \otimes \alpha)((1 \otimes \sigma) \otimes (1 \otimes \sigma))(\alpha^{-1} \otimes \alpha^{-1})\,\Delta(g) \\
&= (\alpha(1 \otimes \sigma)\alpha^{-1}) \otimes (\alpha(1 \otimes \sigma)\alpha^{-1})(g \otimes g) \\
&= \psi(\sigma)(g) \otimes \psi(\sigma)(g).
\end{aligned}$$

Since $G = \{g \in KG \mid \Delta(g) = g \otimes g\}$, we have $\psi(\sigma)(g) \in G$. The fact that α, $1 \otimes \sigma$ are algebra maps implies that $\psi(\sigma)$ is an automorphism of G. Therefore $\psi \in \Psi'(\pi, G)$ and thus the proof of the theorem is completed.

The coalgebra structure of the Hopf algebra $KG^{\psi(\pi)}$ can be determined in

PROPOSITION 1.5. *Let k be a field, let K be a finite Galois extension of k, and let $\pi = \mathrm{Gal}(K/k)$ be the Galois group of K over k. Let G be a finite group, let ψ be a group homomorphism of π to* Aut G, *and let $G = D_1 \cup D_2 \cup \cdots \cup D_n$ be the decomposition of G into $\psi(\pi)$-orbits D_i. Then*

$$(KG^{\psi(\pi)})^* = (KD_1^{\psi(\pi)})^* \oplus \cdots \oplus (KD_n^{\psi(\pi)})^*$$

as k-algebras. Further let g_i be an element of D_i and $\pi_i = \{\tau \in \pi \mid \psi(\tau)(g_i) = g_i\}$. Then $(KD_i^{\psi(\pi)})^* \cong K^{\pi_i}$ as k-algebras.

Proof. In the proof of (1.3), it is shown that $KG^{\psi(\pi)} = \sum^{\oplus} KD_i^{\psi(\pi)}$ as k-vector spaces and $\Delta(KD_i^{\psi(\pi)}) \subseteq KD_i^{\psi(\pi)} \otimes KD_i^{\psi(\pi)}$. From this we see that $(KD)_i^{\psi(\pi)*}$ is a k-algebra and $(KG^{\psi(\pi)})^* = \sum^{\oplus} (KD_i^{\psi(\pi)})^*$ as k-algebras. Hence we only need to prove that $(KD_i^{\psi(\pi)})^* \cong K^{\pi_i}$ as k-algebras. Let $\{\tau_0\ (=1), \tau_1, ..., \tau_l\}$ be a set of representatives of π/π_i in π and $\{t_0, t_1, ..., t_l\}$ a k-basis of K^{π_i}. In the proof of (1.3) we get that $\{a_k = \sum_j \tau_j(t_k)\, \psi(\tau_j)(g_i)\}$ is a k-basis of $KD_i^{\psi(\pi)}$. We define a k-linear map μ; $KD_i^{\psi(\pi)} \to K^{\pi_i}$ by $\mu(a_k) = t_k$ and a multiplication $\circ$ on $(K^{\pi_i})^*$ is induced to make the following diagram commutative:

$$\begin{array}{ccc} (KD_i^{\psi(\pi)})^* \otimes_k (KD_i^{\psi(\pi)})^* & \xrightarrow{\Delta^*} & (KD_i^{\psi(\pi)})^* \\ \uparrow{\scriptstyle \mu^*\otimes\mu^*} & & \uparrow{\scriptstyle \mu^*} \\ (K^{\pi_i})^* \otimes_k (K^{\pi_i})^* & \xrightarrow{\circ} & (K^{\pi_i})^* \end{array}$$

As usual $\mathrm{tr} = \mathrm{tr}_{K^{\pi_i}/k}$ denotes the trace from K^{π_i} to k. For $a \in K^{\pi_i}$ we define a k-linear map $\mathrm{Tr}(a)$: $K^{\pi_i} \to k$ by $\mathrm{Tr}(a)(b) = \mathrm{tr}(ab)$ for all $b \in K^{\pi_i}$. Since $(K^{\pi_i})^* = \{\mathrm{Tr}(a) \mid a \in K^{\pi_i}\}$, we can define a k-linear map ν; $(K^{\pi_i})^* \to K^{\pi_i}$ by $\nu(\mathrm{Tr}(a)) = a$. Let $\eta = \nu\mu^{*-1}$. Now we prove that the k-linear map

$$\eta\colon (KD_i^{\psi(\pi)})^* \to K^{\pi_i}$$

is an isomorphism. By (1.2) we can find $(l+1)^3$ elements α_{ij}^m of k such that $\sum_{i,j} \alpha_{ij}^m \tau_s(t_i)\, \tau_t(t_j) = \tau_s(t_m)\, \delta_{st}$. It has been shown, in the proof of (1.3), that $\Delta(a_m) = \sum_{i,j} \alpha_{ij}^m a_i \otimes a_j$. Then for $a, b \in K^{\pi_i}$,

$$\begin{aligned}
&(\mathrm{Tr}(a) \circ \mathrm{Tr}(b))(t_m) \\
&\quad = (\mu^*)^{-1} \Delta^*(\mu^* \otimes \mu^*)(\mathrm{Tr}(a) \otimes \mathrm{Tr}(b))(t_m) \\
&\quad = \Delta^*(\mu^* \otimes \mu^*)(\mathrm{Tr}(a) \otimes \mathrm{Tr}(b))(a_m) \\
&\quad = (\mathrm{Tr}(a) \otimes \mathrm{Tr}(b))(\mu \otimes \mu)\Big(\sum_{i,j} \alpha_{ij}^m a_i \otimes a_j\Big) \\
&\quad = \sum_{i,j} \alpha_{ij}^m\, \mathrm{tr}(at_i)\, \mathrm{tr}(bt_j) \\
&\quad = \sum_{i,j} \alpha_{ij}^m \sum_{s,t} \tau_s(a)\, \tau_s(t_i)\, \tau_t(b)\, \tau_t(t_j) \\
&\quad = \sum_{s,t} \tau_s(a)\, \tau_t(b)\, \tau_s(t_m)\, \delta_{st} \\
&\quad = \sum_{s} \tau_s(ab)\, \tau_s(t_m) \\
&\quad = \mathrm{tr}(abt_m).
\end{aligned}$$

Therefore $\mathrm{Tr}(a) \circ \mathrm{Tr}(b) = \mathrm{Tr}(ab)$, which implies that $(KD_i^{\psi(\pi)})^* \simeq K^{\pi_i}$ as k-algebras.

2. Simple Components

In this section we use the following notations for $n \times n$ matrices:

I: the identity matrix;

J: the matrix whose ij-component $= 1$ $\quad i - j \equiv 1 \pmod n$
$= 0$ $\quad i - j \not\equiv 1 \pmod n$,

$\mathrm{diag}\{a_1, a_2, \ldots, a_n\}$: the diagonal matrix whose ii-component is a_i.

Our main result in this section is given in

Proposition 2.1. *Let k be a field of characteristic 0. Let K be a finite cyclic Galois extension of k. Then any cyclic k-algebra (K, α), $\alpha \in k - \{0\}$, is realizable as a simple component of a finite-dimensional cocommutative k-Hopf algebra.*

Proof. Let $n = [K : k]$. Let β be an nth root of α^{-1} and let ϵ be a primitive nth root of unity. Write $\pi = \mathrm{Gal}(K(\beta, \epsilon)/k)$, $\pi_0 = \mathrm{Gal}(k(\beta, \epsilon)/k)$, $\pi_1 = \mathrm{Gal}(K(\beta, \epsilon)/k(\beta, \epsilon))$, $\pi_2 = \mathrm{Gal}(K(\beta, \epsilon)/K)$ and put $\mathrm{Gal}(K/k) = \langle u \rangle$. We denote by $\beta\epsilon_1$ $(=\beta)$, $\beta\epsilon_2, \ldots, \beta\epsilon_h$ all distinct elements of $\{\sigma(\beta) \mid \sigma \in \pi_0\}$. Since $\sigma(\beta)$ is an nth root of α^{-1}, ϵ_i is an nth root of unity. To prove the theorem, it suffices to construct a metabelian group G, an irreducible representation ρ of G over $K(\beta, \epsilon)$ of degree n and a group homomorphism ψ of π to Aut G such that $\psi(\sigma)(e) = e$ for all $\sigma \in \pi$ and $(K(\beta, \epsilon)Ge)^{\psi(\pi)} \cong (K, \alpha)$ as a k-algebra, where e is the central idempotent of $K(\beta, \epsilon)G$ corresponding to ρ. In fact, if we can find such G, ρ, and ψ, then by (1.3) $(K(\beta, \epsilon)G)^{\psi(\pi)}$ is a k-Hopf algebra and (K, α) is a simple component of $(K(\beta, \epsilon)G)^{\psi(\pi)}$.

We denote by G the metabelian group generated by $x_1, \ldots, x_n, y_1, \ldots, y_h$ with relations $x_i^2 = 1$, $y_s^n = 1$, $x_ix_j = x_jx_i$, $y_sy_t = y_ty_s$, $y_s^{-1}x_1y_s = x_2, \ldots,$ $y_s^{-1}x_ny_s = x_1$ for all $1 \leqslant i, j \leqslant n$ and all $1 \leqslant s, t \leqslant h$. We define a $K(\beta, \epsilon)$-algebra homomorphism ρ of $K(\beta, \epsilon)G$ to the full matrix algebra $M_n(K(\beta, \epsilon))$ of degree n by

$$\rho(x_i) = \mathrm{diag}\{1, \ldots, 1, \overset{i}{-1}, 1, \ldots, 1\}$$

and

$$\rho(y_i) = \epsilon_i^{-1}({}^tJ).$$

Then ρ is an epimorphism, and therefore it is an irreducible representation of G.

For $\sigma \in \pi$ we define a permutation $\psi(\sigma)$ of $\{x_1, \ldots, x_n, y_1, \ldots, y_h\}$ as follows:

(1) If $\sigma\pi_2 = u^l$, then $\psi(\sigma)(x_i) = x_{i+l}$, $i = 1, \ldots, n$, where subscripts are added mod n.

(2) If $\sigma(\beta\epsilon_i) = \beta\epsilon_j$, then $\psi(\sigma)(y_i) = y_j$.

By the definition of G $\psi(\sigma)$ can be regarded as an automorphism of G. So we get a group homomorphism $\psi\colon \pi \to \operatorname{Aut} G$.

Let e be the central idempotent of $K(\beta, \epsilon)G$ corresponding to ρ. Let χ be the character of ρ. Then it is well known that $e = \chi(1) \mid G \mid^{-1} \sum_{g\in G} \chi(g^{-1})g$. If we can show

$$\sigma\chi = \chi\psi(\sigma)$$

then

$$\begin{aligned}
&\psi(\sigma)(e)\\
&\quad = \psi(\sigma)(\chi(1) \mid G \mid^{-1} \sum_{g\in G} \chi(g^{-1})g)\\
&\quad = \chi(1) \mid G \mid^{-1} \sum_{g\in G} \sigma(\chi(g^{-1}))\, \psi(\sigma)(g)\\
&\quad = \chi(1) \mid G \mid^{-1} \sum_{g\in G} \chi((\psi(\sigma)(g))^{-1})\, \psi(\sigma)(g)\\
&\quad = \chi(1) \mid G \mid^{-1} \sum_{g\in G} \chi(g^{-1})g\\
&\quad = e.
\end{aligned}$$

Therefore, in order to show that $\psi(\sigma)(e) = e$, we only need to show that

$$\begin{aligned}
&\sigma(\chi(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_1^{f_1}y_2^{f_2} \cdots y_h^{f_h})\\
&\quad = \chi(\psi(\sigma)(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_1^{f_1}y_2^{f_2} \cdots y_h^{f_h})).
\end{aligned}$$

By the definition of ρ we have

$$\begin{aligned}
&\chi(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_1^{f_1}y_2^{f_2} \cdots y_h^{f_h})\\
&\quad = \operatorname{Tr}(\operatorname{diag}\{(-1)^{e_1}, (-1)^{e_2},\ldots, (-1)^{e_n}\}({}^tJ)^{f_1+f_2+\cdots+f_h}\epsilon_1^{-f_1}\epsilon_2^{-f_2} \cdots \epsilon_h^{-f_h})\\
&\quad = \epsilon_1^{-f_1}\epsilon_2^{-f_2} \cdots \epsilon_h^{-f_h}((-1)^{e_1} + \cdots + (-1)^{e_n}) \qquad \text{when} \quad n \mid f_1 + \cdots + f_h\,,\\
&\quad = 0 \qquad \text{when} \quad n \nmid f_1 + \cdots + f_h\,.
\end{aligned}$$

Let $x_{i_s} = \psi(\sigma)(x_s)$ and $y_{j_s} = \psi(\sigma)(y_s)$. Then, in the case where $n \mid f_1 + \cdots + f_h$,

$$\begin{aligned}
&\beta^{-(f_1+\cdots+f_h)}\epsilon_{j_1}^{-f_1} \cdots \epsilon_{j_h}^{-f_h}\\
&\quad = (\beta\epsilon_{j_1})^{-f_1} \cdots (\beta\epsilon_{j_h})^{-f_h}\\
&\quad = (\sigma(\beta\epsilon_1))^{-f_1} \cdots (\sigma(\beta\epsilon_h))^{-f_h}\\
&\quad = \sigma(\beta^{-(f_1+\cdots+f_h)})(\sigma(\epsilon_1))^{-f_1} \cdots (\sigma(\epsilon_h))^{-f_h}\\
&\quad = \beta^{-(f_1+\cdots+f_h)}(\sigma(\epsilon_1))^{-f_1} \cdots (\sigma(\epsilon_h))^{-f_h}
\end{aligned}$$

which implies

$$\epsilon_{j_1}^{-f_1} \cdots \epsilon_{j_h}^{-f_h} = (\sigma(\epsilon_1))^{-f_1} \cdots (\sigma(\epsilon_h))^{-f_h}.$$

Hence we have

$$\begin{aligned}
&\chi(\psi(\sigma)(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_1^{f_1}y_2^{f_2} \cdots y_h^{f_h})) \\
&\quad = \chi(x_{i_1}^{e_1}x_{i_2}^{e_2} \cdots x_{i_n}^{e_n}y_{j_1}^{f_1}y_{j_2}^{f_2} \cdots y_{j_h}^{f_h}) \\
&\quad = \epsilon_{j_1}^{-f_1}\epsilon_{j_2}^{-f_2} \cdots \epsilon_{j_2}^{-f_h}((-1)^{e_1} + \cdots + (-1)^{e_n}) \qquad \text{when} \quad n \mid f_1 + \cdots + f_h \\
&\quad = 0 \qquad \text{when} \quad n \nmid f_1 + \cdots + f_h \\
&\quad = (\sigma(\epsilon_1))^{-f_1} \cdots (\sigma(\epsilon_h))^{-f_h}((-1)^{e_1} + \cdots + (-1)^{e_n}) \qquad \text{when} \quad n \mid f_1 + \cdots + f_h \\
&\quad = 0 \qquad \text{when} \quad n \nmid f_1 + \cdots + f_h \\
&\quad = \sigma(\chi(x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}y_1^{f_1}y_2^{f_2} \cdots y_h^{f_h})).
\end{aligned}$$

It remains to be shown that $(K(\beta, \epsilon)Ge)^{\psi(\pi)} \cong (K, \alpha)$ as k-algebras. Let

$$\begin{aligned}
z_i &= 2^{-1}(1 - x_i), \\
\hat{K} &= \{(az_1 + u(a)z_2 + \cdots + u^{n-1}(a)z_n)e \mid a \in K\}
\end{aligned}$$

and $\hat{u} = |\pi|^{-1} \sum_{\sigma\in\pi} \sigma(\beta^{-1})\, y_{i_\sigma}^{-1}\, e$, where i_σ is defined by $\sigma(\beta) = \beta\epsilon_{i_\sigma}$. Then if $\sigma \in \pi_2$,

$$\begin{aligned}
&\psi(\sigma)((az_1 + u(a)z_2 + \cdots + u^{n-1}(a)z_n)e) \\
&\qquad = (az_1 + u(a)z_2 + \cdots + u^{n-1}(a)z_n)e,
\end{aligned}$$

and if $\sigma\pi_2 = u$,

$$\begin{aligned}
&\psi(\sigma)((az_1 + u(a)z_2 + \cdots + u^{n-1}(a)z_n)e) \\
&\qquad = (u(a)z_2 + u^2(a)z_3 + \cdots + u^n(a)z_1)e \\
&\qquad = (az_1 + u(a)z_2 + \cdots + u^{n-1}(a)z_n)e,
\end{aligned}$$

which implies $\hat{K} \subseteq (K(\beta, \epsilon)Ge)^{\psi(\pi)}$. For each $\tau \in \pi$,

$$\begin{aligned}
&\psi(\tau)(\hat{u}) \\
&\qquad = |\pi|^{-1} \sum_{\sigma\in\pi} \tau(\sigma(\beta^{-1}))\, \psi(\tau)(y_{i_\sigma}^{-1})e \\
&\qquad = |\pi|^{-1} \sum_{\sigma\in\pi} \tau\sigma(\beta^{-1})\, y_{i_{\tau\sigma}}^{-1}e \\
&\qquad = |\pi|^{-1} \sum_{\sigma\in\pi} \sigma(\beta^{-1})\, y_{i_\sigma}^{-1}e \\
&\qquad = \hat{u},
\end{aligned}$$

because $\tau\sigma(\beta) = \tau(\beta\epsilon_{i_\sigma}) = \beta\epsilon_{i_{\tau\sigma}}$ means $\psi(\tau)(y_{i_\sigma}) = y_{i_{\tau\sigma}}$. Therefore $\hat{u} \in (K(\beta, \epsilon)Ge)^{\psi(\pi)}$.

On the other hand,

$$\rho(\hat{K}) = \{\mathrm{diag}\{a, u(a),..., u^{n-1}(a)\} \mid a \in K\}$$

and

$$\begin{aligned}\rho(\hat{u}) &= |\pi|^{-1} \sum_{\sigma\in\pi} \beta^{-1}\epsilon_{i_\sigma}^{-1}(\epsilon_{i_\sigma}J)\\ &= \beta^{-1}J.\end{aligned}$$

Since $(\beta^{-1}J)^n = \beta^{-n}I = \alpha I$ and

$$\begin{aligned}&(\beta^{-1}J)^{-1}\,\mathrm{diag}\{a, u(a),..., u^{n-1}(a)\}\,\beta^{-1}J\\ &= \mathrm{diag}\{u(a),..., u^{n-1}(a), a\},\end{aligned}$$

we have $\hat{K} \cong K$ by the correspondence $(\sum_{i=0}^{n-1} u^i(a)\, z_{i+1})e \leftrightarrow a$ and $\hat{K} + \hat{K}\hat{u} + \cdots + \hat{K}\hat{u}^{n-1} \cong (K, \alpha)$ by the correspondence $(\sum_{i=0}^{n-1} u^i(a)z_{i+1})e \leftrightarrow a$, $\hat{u} \leftrightarrow u$. Accordingly $\dim_k(\hat{K} + \hat{K}\hat{u} + \cdots + \hat{K}\hat{u}^{n-1}) = \dim_k(K, \alpha) = n^2 = \dim_k(K(\beta, \epsilon)Ge)^{\psi(\pi)}$, which implies

$$\hat{K} + \hat{K}\hat{u} + \cdots + \hat{K}\hat{u}^{n-1} = (K(\beta, \epsilon)Ge)^{\psi(\pi)}.$$

Thus $(K(\beta, \epsilon)Ge)^{\psi(\pi)} \cong (K, \alpha)$. This completes the proof of the proposition.

We have, in the commutative case,

PROPOSITION 2.2. *Let k be a field of characteristic* 0. *Let $L_1, L_2, ..., L_t$ be finite extensions of k. Then there exists a finite-dimensional commutative cocommutative k-Hopf algebra H of which $L_1 \oplus \cdots \oplus L_t$ is a direct summand as a k-algebra.*

Proof. Let K be a finite Galois extension of k containing $L_1, ..., L_t$. Let $\pi = \mathrm{Gal}(K/k)$, $\pi_i = \mathrm{Gal}(K/L_i)$ and $\pi/\pi_i = \{\tau_0^{(i)}\pi_i, \tau_1^{(i)}\pi_i, ..., \tau_{n_i}^{(i)}\pi_i\}$ with $\tau_0^{(i)} = 1$. We denote by G the elementary Abelian 2-group generated by $x_j^{(i)}$, $0 \leqslant i \leqslant t$, $0 \leqslant j \leqslant n_i$ with relations $(x_j^{(i)})^2 = 1$, $x_j^{(i)}x_l^{(k)} = x_l^{(k)}x_j^{(i)}$ for all $0 \leqslant i, k \leqslant t$, $0 \leqslant j \leqslant n_i$, $0 \leqslant l \leqslant n_k$. For $\sigma \in \pi$ we define $\psi(\sigma)(x_j^{(i)}) = x_k^{(i)}$ if $\sigma\tau_j^{(i)}\pi_i = \tau_k^{(i)}\pi_i$. As is seen easily, ψ is a group homomorphism of π to Aut G and $D_i = \{x_1^{(i)}, ..., x_{n_i}^{(i)}\}$ is a $\psi(\pi)$-orbit of G. Let $H = \mathrm{Hom}_k(KG^{\psi(\pi)}, k)$. From (1.3) $KG^{\psi(\pi)}$ is a commutative cocommutative k-Hopf algebra, which implies H is a commutative cocommutative k-Hopf algebra. Further, by (1.5), $(KD_1^{\psi(\pi)})^* \oplus \cdots \oplus (KD_t^{\psi(\pi)})^*$ is a direct summand of H as a k-algebra, and $(KD_i^{\psi(\pi)})^* \cong K^{\pi_i} = L_i$, because $\{\tau \in \pi \mid \psi(\tau)(x_0^{(i)}) = x_0^{(i)}\} = \{\tau \in \pi \mid \tau\pi_i = \pi_i\} = \pi_i$.

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